Not-A-Knot Cubic Spline Interpolation

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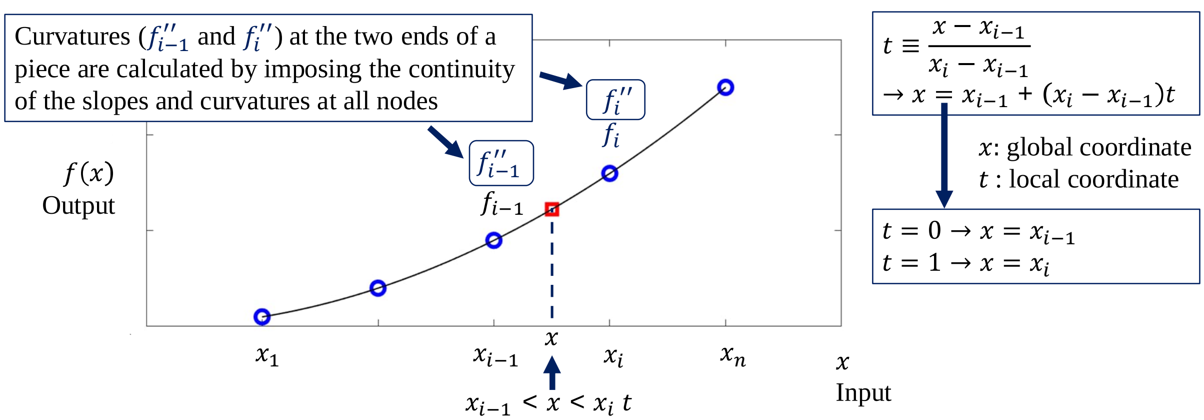
Not-A-Knot Cubic Spline

Formulas

The equation (Eq.1) below is the cubic spline function for interpolating the output f at an input x within the range of a piece with endpoints x_{i-1} and x_i such that $x_{i-1} < x < x_i$. This function uses four known variables $x_{i-1}, x_i, f_{i-1}, f_i$, and two unknown variables f''_{i-1}, f''_i. Even though the two unknown variables f''_{i-1} and f''_i are the curvatures at the two endpoints (x_i and x_{i-1}) of the piece, they are calculated (globally) by imposing the continuity of the slopes and curvatures at all data points (nodes). In practice, the f''_{i-1} and f''_i are calculated by solving the system of equations in (Eq.2)

$$f(x) = \langle 1 \quad t \quad t^2 \quad t^3 \rangle_{1\times 4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}_{4\times 4} \begin{Bmatrix} f_{i-1} \\ f_i \\ f''_{i-1}\,(x_i - x_{i-1})^2 \\ f''_i\,(x_i - x_{i-1})^2 \end{Bmatrix}_{4\times 1} \qquad (1)$$

where t is related to x as $t \equiv \frac{x - x_{i-1}}{x_i - x_{i-1}}$



The two unknown curvatures f''_{i-1} and f''_i are calculated by solving a system of equations in (Eq.2) for a set of n nodes.

$$\begin{bmatrix} x_2 - x_3 & x_3 - x_1 & x_1 - x_2 & \cdots & 0 & 0 & 0 \\ & & & & & & 0 \\ \vdots & 0 & a_{i,i-1} & a_{i,i} & a_{i,i+1} & 0 & \vdots \\ 0 & & & & & & \\ 0 & 0 & 0 & \cdots & x_{n-1} - x_n & x_n - x_{n-2} & x_{n-2} - x_{n-1} \end{bmatrix}_{n\times n} \begin{Bmatrix} f''_1 \\ \vdots \\ f''_i \\ \vdots \\ f''_n \end{Bmatrix}_{n\times 1} = \begin{Bmatrix} 0 \\ \vdots \\ b_i \\ \vdots \\ 0 \end{Bmatrix}_{n\times 1} \qquad (2)$$

where i is index of rows and

$$a_{i,i-1} = \frac{1}{6}(x_i - x_{i-1});\ a_{i,i} = \frac{1}{3}(x_{i+1} - x_{i-1});\ a_{i,i+1} = \frac{1}{6}(x_{i+1} - x_i);\ b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

In the system of equations (Eq.2), the first and the last rows are formulated based on the "NOT-A-KNOT" boundary conditions: the third derivative (f_1''') of the first node uses the third derivative (f_2''') of the second node; the third derivative (f_n''') of the last node uses the third derivative of the second to last node (f_{n-1}''') as

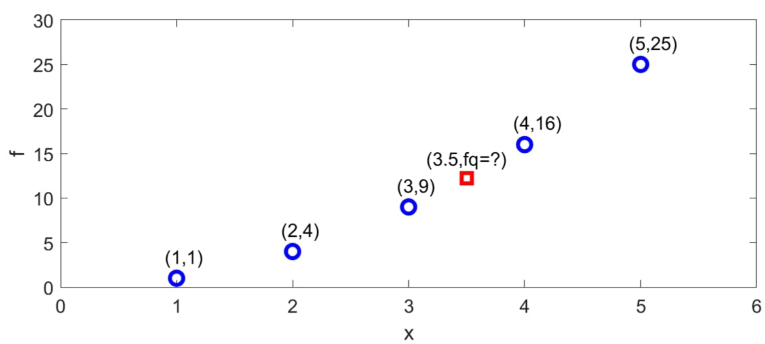
Left End: $f_1''' = f_2''' \quad \rightarrow (x_2 - x_3)f_1'' + (x_3 - x_1)f_2'' + (x_1 - x_2)f_3'' = 0$

Right End: $f_{n-1}''' = f_n''' \quad \rightarrow (x_{n-1} - x_n)f_{n-2}'' + (x_n - x_{n-2})f_{n-1}'' + (x_{n-2} - x_{n-1})f_n'' = 0$

Example

Given: 5 data points: $(x, f) = (1,1), (2,4), (3,9), (4,16), (5,25)$

Interpolate the output f_q at input $x_q = 3.5$ using the natural cubic spline.



Solution

Step 1: Construct matrix $[a]_{n\times n}$ and vector $\{b\}_{n\times 1}$ using Eq. 2

$$[a]_{n\times n} = \begin{bmatrix} x_2 - x_3 & x_3 - x_1 & x_1 - x_2 & \dots & 0 & 0 & 0 \\ & & & & & & 0 \\ \vdots & 0 & a_{i,i-1} & a_{i,i} & a_{i,i+1} & 0 & \vdots \\ 0 & & & & & & \\ 0 & 0 & 0 & \dots & x_{n-1} - x_n & x_n - x_{n-2} & x_{n-2} - x_{n-1} \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & -1 & 2 & -1 \end{bmatrix}_{5\times 5}$$

$$\{b\}_{n\times 1} = \begin{Bmatrix} 0 \\ \vdots \\ b_i \\ \vdots \\ 0 \end{Bmatrix}_{n\times 1} = \begin{Bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 0 \end{Bmatrix}_{5\times 1}$$

where

$$a_{i,i-1} = \frac{1}{6}(x_i - x_{i-1});\ a_{i,i} = \frac{1}{3}(x_{i+1} - x_{i-1});\ a_{i,i+1} = \frac{1}{6}(x_{i+1} - x_i);\ b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

Step 2: Calculate the vector $\{f''\}_{n\times 1}$ using Eq. 2

$$\begin{Bmatrix} f_1'' \\ \vdots \\ f_i'' \\ \vdots \\ f_n'' \end{Bmatrix}_{n\times 1}$$

$$= \begin{bmatrix} x_2 - x_3 & x_3 - x_1 & x_1 - x_2 & \dots & 0 & 0 & 0 \\ & & & & & & 0 \\ \vdots & 0 & a_{i,i-1} & a_{i,i} & a_{i,i+1} & 0 & \vdots \\ 0 & & & & & & \\ 0 & 0 & 0 & \dots & x_{n-1} - x_n & x_n - x_{n-2} & x_{n-2} - x_{n-1} \end{bmatrix}_{n\times n}^{-1} \begin{Bmatrix} 0 \\ \vdots \\ b_i \\ \vdots \\ 0 \end{Bmatrix}_{n\times 1}$$

$$= \begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & -1 & 2 & -1 \end{bmatrix}_{5\times 5}^{-1} \begin{Bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 0 \end{Bmatrix}_{5\times 1} = \begin{Bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{Bmatrix}_{5\times 1}$$

Step 3: Search for the index i of the right endpoint x_i of the piece such that $x_{i-1} < x < x_i$. The left endpoint of the piece is x_{i-1}.

```
for i = 2:n % i is the index of a data point
    % For interpolation: x(i-1)<xq && xq <= x(i)
    % For extrapolation: xq <= x(1) || xq > x(n)
      if xq <= x(1) || xq > x(n)
          break
     end
end
```

Step 4: Calculate t, then construct the vector $= \langle t\rangle_{1\times 4}$ in Eq. 1

$$t = \frac{x - x_{i-1}}{x_i - x_{i-1}} = \frac{4.5-4}{4-5} = 0.5$$

$$\langle t\rangle_{1\times 4} = \langle 1 \quad t \quad t^2 \quad t^3\rangle_{1\times 4} = \langle 1 \quad 0.5 \quad 0.5^2 \quad 0.5^3\rangle_{1\times 4}$$

Step 5: Calculate f_q using Eq.1

$$f_q = f(x = 3.5) = \langle 1 \quad t \quad t^2 \quad t^3\rangle_{1\times 4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}_{4\times 4} \begin{Bmatrix} f_{i-1} \\ f_i \\ f_{i-1}''(x_i - x_{i-1})^2 \\ f_i''(x_i - x_{i-1})^2 \end{Bmatrix}_{4\times 1}$$

$$= \langle 1 \quad 0.5 \quad 0.5^2 \quad 0.5^3\rangle_{1\times 4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}_{4\times 4} \begin{Bmatrix} 9 \\ 16 \\ 2 \\ 2 \end{Bmatrix}_{4\times 1} = 12.2500$$

Code

```
function VID_001_Not_A_Knot_Cubic_Spline
% Author: Hejie Lin
% Date: 2024/11/23
clear all; clf
% Data
data = [...
    1 , 1;...
    2   , 4;...
    3   , 9;...
    4   , 16;...
    5   , 25];
x = data(:,1);
f = data(:,2);
% New known point
xq = 3.5;
% Construct matrix a and vector b
n = size(data,1); % Number of nodes
a = zeros(n,n);
b = zeros(n,1);
for i = 2:n-1 % Index of the row
    a(i,i-1) = (1/6)*(x(i)-x(i-1));
    a(i,i)   = (1/3)*(x(i+1)-x(i-1));
    a(i,i+1) = (1/6)*(x(i+1)-x(i));
    b(i,1)   = (f(i+1)-f(i))/(x(i+1)-x(i))-...
        (f(i-1)-f(i))/(x(i-1)-x(i));
end
% Not-A-Knot Boundary Conditions
%   (x_2-x_3)f"_1+    (x_3-x_1)f"_2+      (x_1-x_2)f"_3 = 0
% (x_n-1-x_n)f"_n-2+(x_n-x_n-2)f"_n-1+(x_n-2-x_n-1)f"_n = 0
a(1,1) = x(2)-x(3);
a(1,2) = x(3)-x(1);
a(1,3) = x(1)-x(2);
a(n,n-2) = x(n-1)-x(n);
a(n,n-1) = x(n)-x(n-2);
a(n,n)   = x(n-2)-x(n-1);
% Calculate f"
f2 = inv(a)*b;
% Constant matrix
m_spline = [...
    1 , 0, 0,    0;...
    -1, 1, -1/3, -1/6;...
    0,  0, 1/2,  0;...
    0,  0, -1/6, 1/6];
% Search for the piece with endpoints x_i-1 and x_i
%     such that x_i-1 < xq <= x_i
for i = 2:n % Indix of nodes
    % For interpolation: x(i-1)<xq && xq <= x(i)
    % For extrapolation: xq <= x(1) || xq > x(n)
    if xq <= x(i) || i == n
        break
    end
end
% Construct vector t and vector f
t = (xq-x(i-1))/(x(i)-x(i-1));
v_t = [1,t,t^2,t^3];
v_f = [f(i-1);f(i);...
    f2(i-1)*(x(i)-x(i-1))^2;f2(i)*(x(i)-x(i-1))^2];
% Calculate fq (new unknown point)
fq = v_t*m_spline*v_f
% Show the results
plot(x,f,'o-k',xq,fq,'sr');
end
```

Last Updated 2024/11/23