Natural Cubic Spline Interpolation

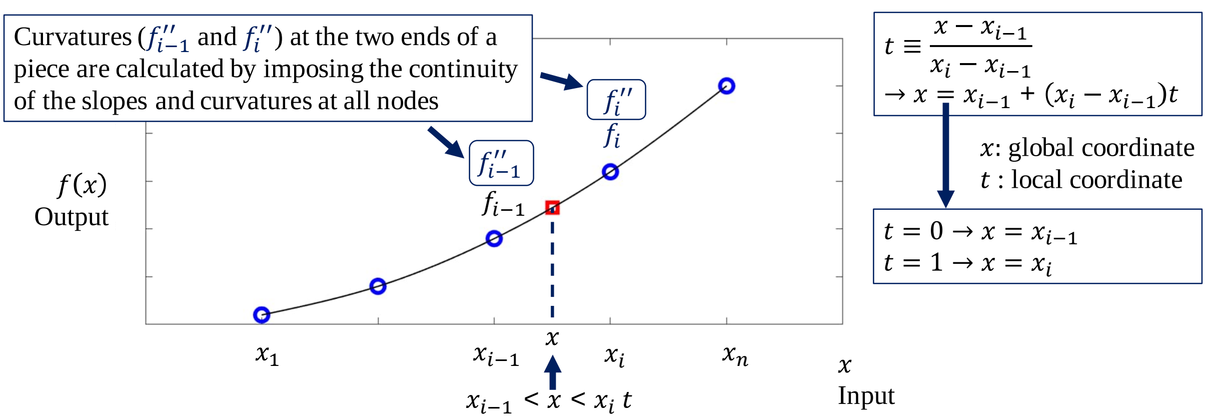
Natural Cubic Spline

Formulas

The equation (Eq.1) below is the cubic spline function for interpolating the output f at an input x within the range of a piece with endpoints x_{i-1} and x_i such that $x_{i-1} < x < x_i$. This function uses four known variables $x_{i-1}, x_i, f_{i-1}, f_i$, and two unknown variables f''_{i-1}, f''_i. Even though the two unknown variables f''_{i-1} and f''_i are the curvatures at the two endpoints (x_i *and* x_{i-1}) of the piece, they are calculated (globally) by imposing the continuity of the slopes and curvatures at all data points (nodes). In practice, the f''_{i-1} and f''_i are calculated by solving the system of equations in (Eq.2)

$$f(x) = \langle 1 \quad t \quad t^2 \quad t^3 \rangle_{1\times 4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}_{4\times 4} \begin{Bmatrix} f_{i-1} \\ f_i \\ f''_{i-1}\,(x_i - x_{i-1})^2 \\ f''_i\,(x_i - x_{i-1})^2 \end{Bmatrix}_{4\times 1} \qquad (1)$$

where t is related to x as $t \equiv \frac{x - x_{i-1}}{x_i - x_{i-1}}$



The two unknown curvatures f''_{i-1} and f''_i are calculated by solving a system of equations in (Eq.2) for a set of n nodes.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & & 0 \\ \vdots & 0 & a_{i,i-1} & a_{i,i} & a_{i,i+1} & 0 & \vdots \\ 0 & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{n\times n} \begin{Bmatrix} f''_1 \\ \vdots \\ f''_i \\ \vdots \\ f''_n \end{Bmatrix}_{n\times 1} = \begin{Bmatrix} 0 \\ \vdots \\ b_i \\ \vdots \\ 0 \end{Bmatrix}_{n\times 1} \qquad (2)$$

where i is index of rows and

$$a_{i,i-1} = \frac{1}{6}(x_i - x_{i-1});\ a_{i,i} = \frac{1}{3}(x_{i+1} - x_{i-1});\ a_{i,i+1} = \frac{1}{6}(x_{i+1} - x_i);\ b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

In the system of equations (Eq.2), the first and the last rows are formulated based on the "NATURAL" boundary conditions (zero curvatures) at the left and right end points as

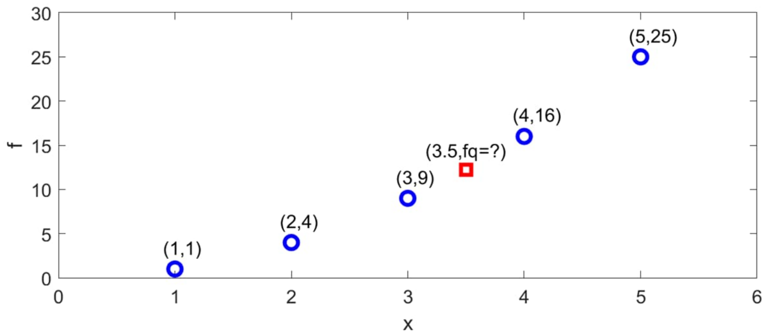
Left End: $f''_1 = 0$

Right End: $f''_n = 0$

Example

Given: 5 data points: $(x, f) = (1,1), (2,4), (3,9), (4,16), (5,25)$

Interpolate the output f_q at input $x_q = 3.5$ using the natural cubic spline.



Solution

Step 1: Construct matrix $[a]_{n\times n}$ and vector $\{b\}_{n\times 1}$ using Eq. 2

$$[a]_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & & 0 \\ \vdots & 0 & a_{i,i-1} & a_{i,i} & a_{i,i+1} & 0 & \vdots \\ 0 & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5}$$

$$\{b\}_{n\times 1} = \begin{Bmatrix} 0 \\ \vdots \\ b_i \\ \vdots \\ 0 \end{Bmatrix}_{n\times 1} = \begin{Bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 0 \end{Bmatrix}_{5\times 1}$$

$$a_{i,i-1} = \frac{1}{6}(x_i - x_{i-1});\ a_{i,i} = \frac{1}{3}(x_{i+1} - x_{i-1});\ a_{i,i+1} = \frac{1}{6}(x_{i+1} - x_i);\ b_i = \frac{f_{i+1}-f_i}{x_{i+1}-x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

Step 2: Calculate the vector $\{f''\}_{n\times 1}$ using Eq. 2

$$\begin{Bmatrix} f_1'' \\ \vdots \\ f_i'' \\ \vdots \\ f_n'' \end{Bmatrix}_{n\times 1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & & 0 \\ \vdots & 0 & a_{i,i-1} & a_{i,i} & a_{i,i+1} & 0 & \vdots \\ 0 & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}_{n\times n}^{-1} \begin{Bmatrix} 0 \\ \vdots \\ b_i \\ \vdots \\ 0 \end{Bmatrix}_{n\times 1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5}^{-1} \begin{Bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 0 \end{Bmatrix}_{5\times 1} = \begin{Bmatrix} 0 \\ 2.5714 \\ 1.7143 \\ 2.5714 \\ 0 \end{Bmatrix}_{5\times 1}$$

Step 3: Search for the index i of the right endpoint x_i of the piece such that $x_{i-1} < x < x_i$. The left endpoint of the piece is x_{i-1}.

```
for i = 2:n % Indix of nodes
    % For interpolation: x(i-1)<xq && xq <= x(i)
    % For extrapolation: xq <= x(1) || xq > x(n)
    if xq <= x(i) || i == n
        break
    end
end
```

Step 4: Calculate t, then construct the vector $= \langle t \rangle_{1\times4}$ in Eq. 1

$$t = \frac{x - x_{i-1}}{x_i - x_{i-1}} = \frac{4.5-4}{4-5} = 0.5$$

$$\langle t \rangle_{1\times4} = \langle 1 \quad t \quad t^2 \quad t^3 \rangle_{1\times4} = \langle 1 \quad 0.5 \quad 0.5^2 \quad 0.5^3 \rangle_{1\times4}$$

Step 5: Calculate f_q using Eq.1

$$f(x) = \langle 1 \quad t \quad t^2 \quad t^3 \rangle_{1\times4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}_{4\times4} \begin{Bmatrix} f_{i-1} \\ f_i \\ f''_{i-1}\,(x_i - x_{i-1})^2 \\ f''_i\,(x_i - x_{i-1})^2 \end{Bmatrix}_{4\times1}$$

$$= \langle 1 \quad 0.5 \quad 0.5^2 \quad 0.5^3 \rangle_{1\times4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}_{4\times4} \begin{Bmatrix} 9.0000 \\ 16.0000 \\ 1.7143 \\ 2.5714 \end{Bmatrix}_{4\times1} = 12.2321$$

Code

```
function cm_natural_cubic_spline
% Author: Hejie Lin
% Reference: Compact Math Series: Not-A-Knot Cubic Spline
clear all; clf
% Data
data = [...
    1 , 1;...
    2   , 4;...
    3   , 9;...
    4   , 16;...
    5   , 25];
x = data(:,1);
f = data(:,2);
% New known point
xq = 3.5;
% Construct matrix a and vector b
n = size(data,1); % Number of nodes
a = zeros(n,n);
b = zeros(n,1);
for i = 2:n-1 % Index of the row
    a(i,i-1) = (1/6)*(x(i)-x(i-1));
    a(i,i)   = (1/3)*(x(i+1)-x(i-1));
    a(i,i+1) = (1/6)*(x(i+1)-x(i));
```

```
    b(i,1)    = (f(i+1)-f(i))/(x(i+1)-x(i))-...
        (f(i-1)-f(i))/(x(i-1)-x(i));
end
% Natural Boundary Conditions
% f"_1 = 0
% f"_n = 0
a(1,1) = 1;
a(n,n) = 1;
% Calculate f"
f2 = inv(a)*b;
% Constant matrix
m_spline = [...
    1 ,  0,    0,    0;...
    -1,  1, -1/3, -1/6;...
     0,  0,  1/2,    0;...
     0,  0, -1/6, 1/6];
% Search for the piece with endpoints x_i-1 and x_i
%     such that x_i-1 < xq <= x_i
for i = 2:n % Indix of nodes
    % For interpolation: x(i-1)<xq && xq <= x(i)
    % For extrapolation: xq <= x(1) || xq > x(n)
    if xq <= x(i) || i == n
        break
    end
end
% Construct vector t and vector f
t = (xq-x(i-1))/(x(i)-x(i-1));
v_t = [1,t,t^2,t^3];
v_f = [f(i-1);f(i);...
    f2(i-1)*(x(i)-x(i-1))^2;f2(i)*(x(i)-x(i-1))^2];
% Calculate fq (new unknown point)
fq = v_t*m_spline*v_f
% Show the results
plot(x,f,'o-k',xq,fq,'sr');
end
```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