Complex Numbers for Harmonic Functions

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Formulas

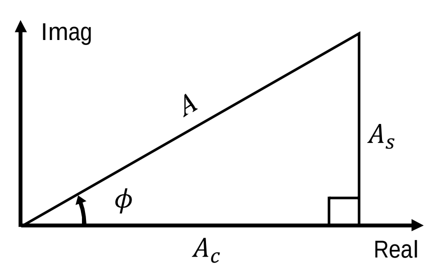
Six equivalent forms (Forms 1-4, 3Re, and 4Re) can be used to describe simple harmonic motion. The following is a summary of the six equivalent forms used to describe simple harmonic motions:

	Real Trigonometric Function	Complex Conjugate Function Pair	Real Part of Complex Function
IP	$A_c \cos(\omega t) - A_s \sin(\omega t)$ Form 1: R_{IP}	$\frac{1}{2}\left[(A_c + jA_s)e^{j\omega t} + (A_c - jA_s)e^{-j\omega t}\right]$ Form 4: C_{IP}	$Re\left[(A_c + jA_s)e^{j\omega t}\right]$ Form 4Re: Re_{IP}
EP	$A\cos(\omega t + \phi)$ Form 2: R_{EP}	$\frac{1}{2}\left[Ae^{j(\omega t+\phi)} + Ae^{-j(\omega t+\phi)}\right]$ Form 3: C_{EP}	$Re\left[Ae^{j(\omega t+\phi)}\right]$ Form 3Re: Re_{EP}

R: Real Trigonometric Function
C: Complex Conjugate Function
IP: Implicit Phase of ϕ
EP: Explicit Phase of ϕ
Re: Real Part

Where A, A_c, and A_s are real numbers and have the relationships shown in the following table. In the figure below, A is a positive number; A_c and A_s are either positive or negative numbers. Therefore the phase angle ϕ is between $-\pi$ and π, and $\phi = \tan^{-1}\left(\frac{A_s}{A_c}\right)$ can be computed by the function atan2(A_s, A_c) in most programming codes.

Geometric Relationship	
Geom 1	$A = \sqrt{A_c^2 + A_s^2}$
Geom 2	$\phi = \tan^{-1}\left(\frac{A_s}{A_c}\right)$
Geom 3	$A_c = A\cos(\phi)$
Geom 4	$A_s = A\sin(\phi)$



Note that Form 1 and Form 2 are formulated using Real numbers (R). Form 3 and Form 4 are formulated using Complex numbers (C). Form 3Re and Form 4Re are formulated using Complex numbers (C) in a simple and short style.

In addition, Form 1 and Form 4 (4Re) are formulated using Implicit Phase (IP) of ϕ. Form 2 and Form 3 (3Re) are formulated using Explicit Phase (EP) of ϕ.

Detail derivation of these six equivalent forms can be found in "Lecture Notes on Acoustics and Noise Control" by Hejie Lin, Turgay Bengisu, Zissimos P. Mourelatos, published in 2021 through Springer

Example 1.1

A simple harmonic motion is expressed in Form 1 by a cosine and a sine function as

(Form 1) $x(t) = \sqrt{3}\cos(5t) + \sin(5t)$

Express this simple harmonic motion in Form 2, Form 3, Form 4, Form 3Re, and Form 4Re to complete the six equivalent Forms as shown below

(Form 1)	$A_c \cos(\omega t) - A_s \sin(\omega t)$	Real Trigonometric Function
(Form 2)	$A \cos(\omega t + \phi)$	Real Trigonometric Function
(Form 3)	$\frac{A}{2}\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right]$	Complex Conjugate Function Pair
(Form 4)	$\frac{1}{2}\left[(A_c + jA_s)e^{j\omega t} + (A_c - jA_s)e^{-j\omega t}\right]$	Complex Conjugate Function Pair
(Form 3Re)	$Re\left[Ae^{j(\omega t+\phi)}\right]$	Real Part of Complex Function
(Form 4Re)	$Re\left[(A_c + jA_s)e^{j\omega t}\right]$	Real Part of Complex Function

Where A, A_c, A_s, and ϕ are real numbers and can be related using the following formula

$$A = \sqrt{A_c^2 + A_s^2};\ \phi = \tan^{-1}\left(\frac{A_s}{A_c}\right);\ A_c = A\cos(\phi);\ A_s = A\sin(\phi)$$

Example 1 Solution

Compare the given simple harmonic motion (Form 1) with the Form 1 formulas

$$x(t) = \sqrt{3}\cos(5t) + \sin(5t) \quad \text{(Form 1:}R_{IP}\text{)}$$

(Form 1) $A_c \cos(\omega t) - A_s \sin(\omega t)$ Real Trigonometric Function

Therefore,

$$\omega = 5$$

$$A_c = \sqrt{3}$$

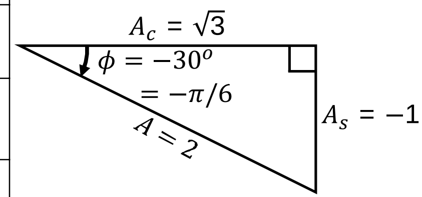
$$A_s = -1$$

then

$$A = \sqrt{A_c^2 + A_s^2} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$$

$$\phi = \tan^{-1}\left(\frac{A_s}{A_c}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}\ \text{[rad] or } \frac{11}{6}\pi\ \text{[rad]}$$

Geometric Relationship	
Geom 1	$A = \sqrt{A_c^2 + A_s^2}$
Geom 2	$\phi = \tan^{-1}\left(\frac{A_s}{A_c}\right)$
Geom 3	$A_c = A\cos(\phi)$
Geom 4	$A_s = A\sin(\phi)$



Therefore, we have

$$x(t) = A\cos(\omega t + \phi)$$

$$= 2\cos\left(5t - \frac{\pi}{6}\right) \qquad \text{(Form 2:}R_{EP}\text{)}$$

$$x(t) = \frac{1}{2}A\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right]$$

$$= \left[e^{j\left(5t-\frac{\pi}{6}\right)} + e^{-j\left(5t-\frac{\pi}{6}\right)}\right] \qquad \text{(Form 3:}C_{EP}\text{)}$$

$$x(t) = \frac{1}{2}\{(A_c + jA_s)e^{j\omega t} + (A_c - jA_s)e^{-j\omega t}\}$$

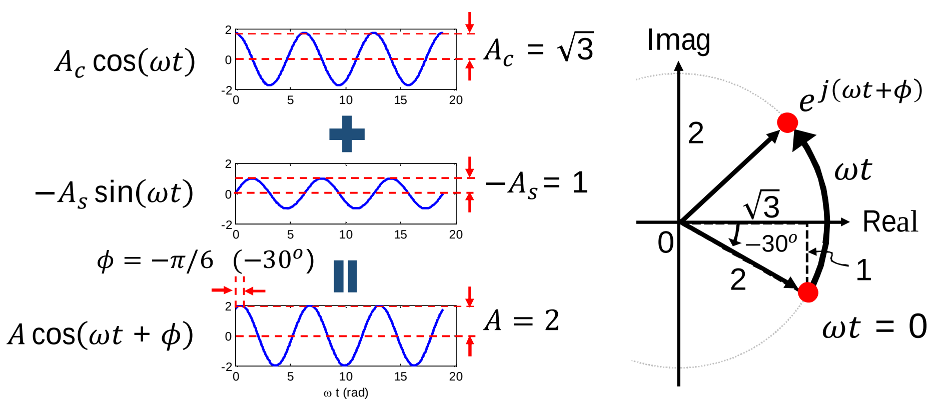
$$= \frac{1}{2}\left[\left(\sqrt{3} - j\right)e^{j(5t)} + \left(\sqrt{3} + j\right)e^{-j(5t)}\right] \qquad \text{(Form 4:}C_{IP}\text{)}$$

$$x(t) = Re\left[Ae^{j(\omega t+\phi)}\right]$$

$$= Re\left[2e^{j\left(5t-\frac{\pi}{6}\right)}\right] \qquad \text{(Form 3Re:}Re_{EP}\text{)}$$

$$x(t) = Re\left[(A_c + jA_s)e^{j\omega t}\right]$$

$$= Re\left[\left(\sqrt{3} - j\right)e^{j(5t)}\right] \qquad \text{(Form 4Re:}Re_{IP}$$



Example 2

A simple harmonic motion is expressed as the real part of a complex function

(Form 4Re) $x(t) = Re\left[(-2 + j3)e^{j4\mathrm{t}}\right]$

Express this simple harmonic motion in the following six equivalent Forms as

(Form 1) $A_c \cos(\omega t) - A_s \sin(\omega t)$ Real Trigonometric Function

(Form 2) $A \cos(\omega t + \phi)$ Real Trigonometric Function

(Form 3) $\frac{A}{2}\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right]$ Complex Conjugate Function Pair

(Form 4) $\frac{1}{2}\left[(A_c + jA_s)e^{j\omega t} + (A_c - jA_s)e^{-j\omega t}\right]$ Complex Conjugate Function Pair

(Form 3Re) $Re\left[Ae^{j(\omega t+\phi)}\right]$ Real Part of Complex Function

(Form 4Re) $Re\left[(A_c + jA_s)e^{j\omega t}\right]$ Real Part of Complex Function

Where A, A_c, A_s, and ϕ can be related using the following formula

$$A = \sqrt{A_c^2 + A_s^2}; \quad \phi = \tan^{-1}\left(\frac{A_s}{A_c}\right); \quad A_c = A\cos(\phi)\,; \; A_s = A\sin(\phi)$$

<u>Example 2 Solution</u>

Compare the given simple harmonic motion (Form 4Re) with the Form 4Re formula

$$x(t) = Re\left[(-2 + j3)e^{j4\mathrm{t}}\right] \qquad \text{(Form 4Re:}Re_{IP}\text{)}$$

(Form 4Re) $Re\left[(A_c + jA_s)e^{j\omega t}\right]$ Real Part of Complex Function

Therefore, we have

$$\omega = 4; A_c = -2; A_s = 3$$

then

$$A = \sqrt{A_c^2 + A_s^2} = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

$$\phi = \tan^{-1}\left(\frac{A_s}{A_c}\right) = \tan^{-1}\left(\frac{3}{-2}\right) = 2.159 \text{ [rad]}$$

Therefore, we have

$$x(t) = A_c \cos(\omega t) - A_s \sin(\omega t)$$

$$= -2\cos(4t) - 3\sin(4t) \qquad \text{(Form 1:}R_{IP}\text{)}$$

$$x(t) = A\cos(\omega t + \phi)$$

$$= \sqrt{13}\cos(4t + 2.159) \qquad \text{(Form 2:}R_{EP}\text{)}$$

$$x(t) = \frac{1}{2}A\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right]$$

$$= \frac{1}{2}\left[\sqrt{13}e^{j(4t+2.159)} + \sqrt{13}e^{-j(4t+2.159)}\right] \qquad \text{(Form 3:}C_{EP}\text{)}$$

$$x(t) = A\left[e^{j(\omega t+\phi)}\right]$$

$$= Re\left[\sqrt{13}e^{j(4t+2.159)}\right] \qquad \text{(Form 3Re:}Re_{EP}\text{)}$$

$$x(t) = \frac{1}{2}\left[(A_c + jA_s)e^{j\omega t} + (A_c - jA_s)e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left\{\left[(-2 + j3)e^{j4t}\right] + (-2 - j3)e^{-j4t}\right\} \qquad \text{(Form 4:}C_{IP}\text{)}$$

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The material in this book is extracted and modified from the first chapter of "Lecture Notes on Acoustics and Noise Control" by Hejie Lin, Turgay Bengisu, Zissimos P. Mourelatos, published in 2021 through Springer

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